Journal of Differential Equations **175**, 244–274 (2001)
doi:10.1006/jdeq.2000.3921, available online at http://www.idealibrary.com on IDEAL®

Marchenko–Ostrovki Mapping for Periodic Zakharov–Shabat Systems

Evgeni Korotyaev[1]

Institut für Mathematik, Humboldt Universität zu Berlin, Rudower Chaussee 25, 12489 Berlin
E-mail: ek@mathematik.hu-berlin.de

Received June 28, 1999

Consider the Zakharov–Shabat (or Dirac) operator T in the Hilbert space $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ with real periodic vector potentials $(q_1, q_2)\in H=L^2(\mathbb{T})\oplus L^2(\mathbb{T})$

Zakharov–Shabat equation with Dirichlet boundary conditions at 0, 1, the square of the height of vertical slits on the quasimomentum domain, and the points on these slits, we construct the Marchenko–Ostrovki (vertical slits) mapping for the periodic Zakharov–Shabat systems $h: H\to \ell^2\oplus\ell^2$. Using nonlinear functional analysis in Hilbert spaces, we show that this mapping is a real analytic isomorphism. In the second part of our paper we prove a new identity for the effective masses.

1. INTRODUCTION

We consider the Zakharov–Shabat (or Dirac) operator T acting on the Hilbert space $\mathscr{H}=L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ and having the form

$$T=J\frac{d}{dx}+V(x),\qquad V\equiv\begin{pmatrix} q_1 & q_2\\ q_2 & -q_1\end{pmatrix}=q_1J_1+q_2J_2,$$

where

$$J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\qquad J_1=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix},\qquad J_2=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},$$

[1] Partially supported by SFB 288 and ISSEP.

0022-0396/01 $35.00

and let $q=(q_1, q_2) \in H=L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$ be a real 1-periodic vector function of $x \in \mathbb{R}$. We also use the Zakharov–Shabat (or Dirac) system for a vector function f

$$Jf' + Vf = zf, \; z \in \mathbb{C}, \qquad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \; x \in \mathbb{R}. \tag{1.1}$$

Here and below $(\,')=\partial/\partial x$, $(\dot{\ })=\partial/\partial z$, $\partial=\partial/\partial q=(\partial_1, \partial_2)=(\partial/\partial q_1, \partial/\partial q_2)$. We recall well known results on the Dirac operator (see [Kr, LS] for details). The boundary value problem (1.1) with the condition $f(0)=f(1)$ is called by periodic and the boundary value problem (1.1) with the condition $f(0)=-f(1)$ is called by antiperiodic. We denote the eigenvalues of the periodic problem by $z_{2n}^{\pm}$ and the eigenvalues of the antiperiodic problem by $z_{2n+1}^{\pm}$, $n \in \mathbb{Z}$. It is well known that $\cdots < z_{2n-1}^{-} \leqslant z_{2n-1}^{+} < z_{2n}^{-} \leqslant z_{2n}^{+} < \cdots$, and

$$z_n^{\pm} = n(\pi + o(1)) \qquad \text{as} \quad |n| \to \infty.$$

We introduce the fundamental solutions of (1.1) (vector-functions)

$$\varphi(x, z, q) = \begin{pmatrix} \varphi_1(x, z, q) \\ \varphi_2(x, z, q) \end{pmatrix}, \qquad \vartheta(x, z, q) = \begin{pmatrix} \vartheta_1(x, z, q) \\ \vartheta_2(x, z, q) \end{pmatrix},$$

satisfying the conditions

$$\varphi(0, z, q) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \vartheta(0, z, q) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and we have the identity

$$\vartheta_1(x, z, q)\, \varphi_2(x, z, q) - \vartheta_2(x, z, q)\, \varphi_1(x, z, q) = 1. \tag{1.2}$$

We introduce the Lyapunov function for the Dirac equation

$$\Delta(z, q) = \frac{\varphi_2(1, z, q) + \vartheta_1(1, z, q)}{2}, \qquad z \in \mathbb{C}, \; q \in H_C.$$

The function $\dot{\Delta}(z, q)$ has a zero z_n in a “closed gap” $[z_n^-, z_n^+]$. Now we define the conformal mapping (the quasimomentum) $k: \mathscr{Z} \to \mathscr{K}$ such that (see [Mi1])

$$k(z) = k(z, q) = \arccos \Delta(z, q), \qquad z \in \mathscr{Z} = \mathbb{C} \setminus \cup \bar{\gamma}_n, \qquad \mathscr{K} = \mathbb{C} \setminus \cup \Gamma_n,$$

$$k(z) = z - \frac{\|q\|^2 + o(1)}{2z}, \qquad z \to i\infty,$$

where an excised slit $\Gamma_n=(\pi n-i|h_n|, \pi n+i|h_n|)$ and the height $|h_n|\geqslant 0$ is defined by Eq. (1.3). *Note that the relations* $k(z_0^\pm)=0$ *give the indexing the gaps* γ_n; *i.e., the function* $k(\cdot)$ *maps the gap* γ_0 *on the vertical slit* $\Gamma_0=(-ih_0, ih_0)$. The spectrum of T is purely absolutely continuous and is given by the set $\bigcup \sigma_n$, where a interval $\sigma_n=[z_{n-1}^+, z_n^-]$. These intervals are separated by gaps $\gamma_n=(z_n^-, z_n^+)$ with the length $|\gamma_n|\geqslant 0$. If a gap γ_n is degenerate, i.e., $\gamma_n=\varnothing$, then the corresponding segments σ_n, σ_{n+1} merge. We have $\Delta(z_n^\pm, q)=(-1)^n$, $n\in\mathbb{Z}$ for $q\in H$. We introduce the Hilbert space

$$\ell^2=\left\{f=\{f_n\}_{\mathbb{Z}}, \|f\|^2=\sum |f_n|^2<\infty\right\}.$$

Let H_C be the complexification of the real Hilbert space H. We denote a complex ball in H_C by $B_C(p,t)=\{q\in H_C: \|q-p\|<t\}$, $p\in H_C$, $t>0$. Introduce the Fourier transformation $\Phi: H_C\to \ell^2_C\oplus \ell^2_C$ by the formulas

$$\hat{f}=\Phi f=\{\hat{f}_n\}, \hat{f}_n=\begin{pmatrix} f_{n1}\\ f_{n2}\end{pmatrix}=\int_0^1 e^{2\pi nxJ} f(x)\,dx=\begin{pmatrix} f_{1(nc)}+f_{2(ns)}\\ -f_{1(ns)}+f_{2(nc)}\end{pmatrix},$$

$$f_{m(nc)}=\int_0^1 f_m(x)\cos 2\pi nx\,dx, \qquad f_{m(ns)}=\int_0^1 f_m(x)\sin 2\pi nx\,dx, m=1,2,$$

$$f(x)=(\Phi^*\hat{f})(x)=\sum e^{-2\pi nxJ}\hat{f}_n.$$

We consider the "Dirichlet problem" for the Dirac operator, i.e., the problem (1.1) with the condition $f_1(0,z)=f_1(1,z)=0$. Let $m_n(q)$ denote the Dirichlet eigenvalue; it is well known that $m_n(q)\in[z_n^-, z_n^+]$. Define the mapping $h: q\to h(q)=\{h_n\}_{n\in\mathbb{Z}}$ from H to $\ell^2\oplus\ell^2$ by the rule $h_n=(h_{n1}, h_{n2})\in\mathbb{R}^2$, where the function $|h_n(q)|^2, q\in H$, is defined by the equation

$$\cosh|h_n(q)|=(-1)^n\,\Delta(z_n(q),q), \qquad q\in H, \tag{1.3}$$

and the components have the form

$$h_{n1}=|\,|h_n|^2-h_{n2}^2\,|^{1/2}\operatorname{sign}(z_n(q)-m_n(q)),$$
$$h_{n2}=-\log((-1)^n\,\varphi_2(1,m_n(q),q)), \qquad n\in\mathbb{Z}.$$

Identity (1.2) and $\varphi_2(1,m_n(q),q)=(-1)^n\exp(-h_{n2})$ yield $\vartheta_1(1,m_n(q),q)=(-1)^n\exp(h_{n2})$ and thus

$$(-1)^n\,\Delta(\mu_n(q),q)=\cosh h_{n2}, \tag{1.4}$$

$$\Delta_0(\mu_n(q),q)\equiv\tfrac{1}{2}(\varphi_2(1,m_n(q),q)-\vartheta_1(1,m_n(q),q))$$
$$=-(-1)^n\sinh h_{n2}, \qquad n\in\mathbb{Z}. \tag{1.5}$$

We have the following identities for the Dirac operator from [KK2]

$$\|q\|^2 \equiv \int_0^1 (q_1(t)^2 + q_2(t)^2)\, dt = \frac{2}{\pi} \int_{\mathbb{R}} v(t+i0)\, dt = \frac{1}{\pi} \iint |z'(k) - 1|^2\, du\, dv, \qquad k = u + iv; \tag{1.6}$$

Introduce the Banach spaces $\ell^p = \{f = \{f_n\}_{\mathbb{Z}}, \|f\|_{(p)}^p \equiv \sum |f_n|^p < \infty\}$, $p \geqslant 1$. In analogy to the notation $O(1/n)$ we use the notation $\ell^d(n)$, $d \geqslant 1$, for an arbitrary sequence of numbers, which is an element of ℓ^d (see [PT]). For instance, $a_n = b_n + \ell^d(n)$ is equivalent to $a_n = b_n + c_n$, $\sum |c_n|^d < \infty$. Introduce the operators $S: \ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$ and $U: H \to H$ by the formulas $(Sf)_n = f_{n+1}$ and $(Uq)(x) = e^{\pi x J} q(x)$. We formulate the main result.

THEOREM 1.1. *The mapping* $h: L^2(0,1) \oplus L^2(0,1) \to \ell^2 \oplus \ell^2$ *is a real analytic isomorphism; the following estimates and the identity are fulfilled:*

$$\tfrac{1}{2} \|q\| \leqslant \|h\| \leqslant 3(1 + \|q\|)^{1/2} \|q\|, \tag{1.7}$$

$$\sigma(U^* T U) = \sigma(T) + \pi, \qquad Sh(q) = h(Uq), \qquad q \in H. \tag{1.8}$$

Moreover, for any $d > 1$ *we have*

$$h_n(q) = \hat{q}_n + \ell^d(n), \tag{1.9}$$

$$d_q h_n(q) = e^{2\pi n x J} + \ell^2(n), \tag{1.10}$$

as $n \to \pm\infty$, *uniformly on* $[0,1] \times B_{\mathbb{C}}(p, \varepsilon_p)$ *for each fixed* $p \in H$, *where* $\varepsilon_p = 4^{-4} e^{-3\|p\|}$.

Remark. (i) From the value $h(q)$ we can compute the height $|h_n|$, sign h_{n1}, and h_{n2} for any $n \in \mathbb{Z}$. Using the conformal mapping (quasimomentum) $k(z)$ we find the spectrum of the Dirac operator, and the positions of the Dirichlet eigenvalues $m_n(q)$. (ii) In Theorem 1.1 we reprove Misura's result [Mi1] by the direct method. (iii) The Gelfand–Levitan–Marchenko equation and a trace formula are not used in the proof. (iv) Estimates (1.7) were proved in [K3]. (v) Identities (1.8) show that if the inverse problem is solved for some $h \in \ell^2 \oplus \ell^2$ then we have the solutions of the inverse problem for each $S^m h$, $m \in \mathbb{Z}$, see (6.10–11).

EXAMPLE. Consider now the simple example of the inverse problem. Assume that only one gap $\gamma_0 = (-1, 1)$ is open and other gaps γ_n, $n \neq 0$, are closed. In this case we have the following Dirac operator

$$Hf = Jf' + Vf, \qquad V = \begin{pmatrix} c & s \\ s & -c \end{pmatrix}, \qquad q_1 = c = \cos 2t,\ q_2 = s = \sin 2t,\ t \in [0, \pi).$$

The corresponding the vector h has the form

$$h=\{h_n\}, \qquad h_0=\begin{pmatrix} c \\ s \end{pmatrix}, \qquad h_n=0, n\neq 0.$$

Now assume that $\gamma_N=(-1+r, 1+r), r=\pi N$, and other gaps $\gamma_n, n\neq N$, are closed for some $N\in\mathbb{Z}$. Then

$$\cdots, z^{\pm}_{N-1}=-\sqrt{1+\pi^2}+r, \qquad m_N=-c+r, z^{\pm}_{N+1}=\sqrt{1+\pi^2}+r, \ \cdots,$$

and by (1.8), we obtain

$$h=\{h_n\}, \qquad h_N=\begin{pmatrix} c \\ s \end{pmatrix}, \qquad h_n=0, n\neq N.$$

Hence using again (1.8) we get the solution of the inverse problem in the case of one gap:

$$V=\begin{pmatrix} \cos 2b & \sin 2b \\ \sin 2b & -\cos 2b \end{pmatrix}, \qquad b=t-\pi Nx.$$

For detailed calculations see Section 6.

Theorem 1.1 generalizes the result of [K2] to the case of the Dirac operator with periodic potential matrices. This generalization requires essentially different proofs, since the technique that was used for studing scalar Hill operator could not be directly generalized to the vector Dirac operator. This induced us to reconsider the original version of the proof (Theorem A); this led us evidently to discover simplifications, and also improvements in the estimates. The proof presented here depends on estimates from [K9].

There are various methods of solving inverse problems. We shortly describe the "direct approach" from [GT2, KK1], based on nonlinear functional analysis. Suppose that H, H_1 are real separable Hilbert spaces. The derivative of a map $f: H\to H_1$ at a point $y\in H$ is a bounded linear map from H into H_1, which we denote by $d_y f$. A map $f: H\to H_1$ is compact on H, if it maps a weakly convergent sequence in H into a strongly convergent sequence in H_1. A map $f: H\to H_1$ is a real analytic isomorphism between H and H_1, if f is bijective and both f and f^{-1} are real analytic maps. Let H_C be the complexification ot he real Hilbert space H. We formulate the key result of the direct method for the Direct operator with the needed modification from [K10].

THEOREM A. *Let H, H_1 be real separable Hilbert spaces equipped with norms $\|\cdot\|, \|\cdot\|_1$. Suppose that a map $f: H\to H_1$ satisfies the following conditions*:

(i) *f is real analytic,*

(ii) *the operator $d_q f$ has an inverse for all $q \in H$,*

(iii) *there is a nondecreasing function $p: [0, \infty) \to [0, \infty)$, $p(0)=0$, such that $\|q\| \leqslant p(\|f(q)\|_1)$ for all $q \in H$,*

(iv) *there exists a basis $\{e_n\}_{n \in \mathbb{Z}}$ of H_1 such that each map $(f(\cdot), e_n)_1 : H \to \mathbb{R}$, $n \in \mathbb{Z}$, is compact,*

(v) *for each $\xi > 0$ the set $\{q : \sum n^2 (f(q), e_n)_1^2 < \xi\}$ is compact.*

Then f is a real analytic isomorphism between H and H_1.

We consider the "physical" case. We define the operator T_0, acting on the Hilbert space $\mathscr{H}$, by the formula

$$T_0 = J \frac{d}{dx} + \rho(x) I + m J_1,$$

where the potential energy $\rho \in L^2(0, 1)$ is the 1-periodic function in $x \in \mathbb{R}$, and $m \neq 0$ is the mass of the particle. In the relativistic quantum theory T_0 is called by the 1-dimensional Dirac operator. Let $k = u + iv = k(z)$ be the quasimomentum for T_0 and let $z(k)$ be the inverse function. With and edge of the non-degenerate gap γ_n, we associate an effective masses by the formula

$$\mu_n^{\pm} = \begin{cases} \dfrac{1}{z''(k(z_n^{\pm}))}, & \text{if} \quad |\gamma_n| \neq 0, \\ 0, & \text{if} \quad |\gamma_n| = 0. \end{cases}$$

It is well known that if $|\gamma_n| \neq 0$, then

$$\pm \mu_n^{\pm} > 0, \qquad \text{and} \qquad z(k) = z_n^{\pm} + \frac{(k - \pi n)^2}{2\mu_n^{\pm}} (1 + o(1)), \quad \text{as} \quad z \to z_n^{\pm}.$$

Note that $\sum(\mu_n^- + \mu_n^+) = 0$, see [KK2], and recall $k = u + iv$.

THEOREM 1.2. *Let $\rho \in L^2(0, 1)$ be a 1-periodic real functions in $x \in \mathbb{R}$ and let $m \neq 0$. Then the following identities are fulfilled,*

$$m^2 = \frac{1}{\pi} \iint_{\mathbb{C}} |z'(k) - 1|^2 \, du \, dv = \frac{2}{\pi} \int_{\mathbb{R}} v(t + i0) \, dt = \frac{1}{2} \sum (\mu_n^- z_n^- + \mu_n^+ z_n^+). \tag{1.11}$$

and the series converge absolutely.

This shows that if ρ changes then the Dirichlet integral does not change!

A great number of papers are devoted to the inverse problem for the 1-dimension differential operator with periodic coefficients. Marchenko and Ostrovski [MO1, MO2] proved the continuous isomorphism of the height slit mapping for the Hill operator. Garnett and Trubowitz [GT1] proved the real analytic isomorphism both of the height slit mapping and of the gap length mapping for the case of even potentials. Kargaev and the author [KK1] reproved the result of Garnet and Trubowitz [GT1] by the direct method. Moreover, they considered other mappings. Note that the proofs by the direct method are short but this approach needs some estimates (see (iii) in Theorem A and (1.7)). The author [K2] reproved the result of [MO1, MO2] by the direct mapping. In order to get the estimates we need the "global quasimomentum" which was introduced into the spectral theory of the Hill operator by Firsova [F] and by Marchenko and Ostrovski [MO1] sumiltaneously. Double-sided estimates for various parameters of the Hill operator (the norm of a periodic potential, effective masses, gap lengths, height of slits, and so on) were obtained in [K2–K4, 7] and for the Dirac operator in [KK2, K2, K5]. Pöschel and Trubowitz [PT] wrote a book concerning the inverse Dirichlet problem. Krein [Kr] obtained various results about the Lyapunov function for the Dirac operator. In [Mi1, Mi2] Misura extended the results of [MO1, MO2] for the periodic Dirac operator.

In this paper the result of Misura [Mi1, Mi2] is reproved and the proof is significantly shorter since the direct method is used. We finish the introduction by briefly explaining how the proof by a direct approach will go, i.e., how we verify conditions (i)–(v) of Theorem A for the mapping g. Remark that the theorem from [KK1] does not work since there exists a big difference between the asymptotics of heights for the Hill operator and ones for the Dirac operator (see (1.8)). Asymptotics for the Hill operator is more convenient than asymptotics for the Dirac operator. Note that in the proof we only use some results from [K1, K9, K10], and estimates (1.7) from [K3]. The checking of (i) is based on the analyticity of the functions $\psi(\cdot, z, q)$, $\Delta(z, q)$ of $z \in \mathbb{C}$, $q \in H_C$. In order to obtain the analyticity of the mapping h we need to prove the real analyticity of mappings $z_n(\cdot)$, $|h_n(\cdot)|^2$. To check (ii), we prove that the Fréchet derivatives of the map is a Fredholm operator of index zero with zero-dimensional kernels and, therefore, is invertible. Here we use a result of Paley–Wiener about entire functions and we prove the following fact: for any fixed $q \in H$ the vectors $\{(d_q h_{cn}), (d_q m_n), n \in \mathbb{Z}\}$ form a basis of H. We emphasize the important role of the entire function $\partial\Delta(z, q)$. The verification of (iii) is based on estimates (1.7) proved in [K3]. To check (iv), we use the compactness of the mappings $z_n(\cdot) : H \to \mathbb{R}$, $m_n(\cdot) : H \to \mathbb{R}$ and the fundamental solutions. The checking of (v) is based on estimates (5.2) given in [K9].

2. PRELIMINARIES

Below we need the simple identities for J, J_1, J_2, V:

$$J^2 = -I, \qquad J_1 J_2 = J, \qquad JJ_1 = -J_2, \qquad JJ_2 = J_1, \tag{2.1}$$

$$e^{zJ} = I\cos z + J\sin z,\ z \in \mathbb{C}, \tag{2.2}$$

$$JV = -VJ. \tag{2.3}$$

Identities (2.2)–(2.3) yield

$$e^{zJ}V = Ve^{-zJ}, \qquad z \in \mathbb{C}. \tag{2.4}$$

The 2×2- matrix valued solution of the equation

$$J\psi' + V\psi = z\psi, \qquad \psi(0, z, q) = I, \qquad z \in \mathbb{C}, \qquad q \in H_C, \tag{2.5}$$

has the representation

$$\psi(x, z, q) = e^{-zxJ} + \int_0^x e^{-zJ(x-t)}\, JV(t)\, \psi(t, z, q)\, dt. \tag{2.6}$$

We construct the solution of this integral equation. It is clear that Eq. (2.6) has the solution as a power series in q. That is,

$$\psi(x, z, q) = \sum_{n \geqslant 0} \psi_n(x, z, q), \tag{2.7}$$

where the functions ψ_n are defined by the relations

$$\psi_0(x, z, q) = e^{-zxJ} = \begin{pmatrix} \cos zx & -\sin zx \\ \sin zx & \cos zx \end{pmatrix}, \tag{2.8}$$

$$\psi_n(x, z, q) = \int_0^x e^{-zJ(x-t)}\, JV(t)\, \psi_{n-1}(t, z, q)\, dt, \qquad n \geqslant 1, \tag{2.9}$$

and here for fixed x, z, the function ψ_n is a multi-linear form on $H_C \times \cdots \times H_C$. Using (2.9) we have

$$\psi_1(x, z, q) = \int_0^x e^{-zJ(x-t)}\, JV(t)\, e^{-ztJ}\, dt = \int_0^x e^{-zJ(x-2t)}\, JV(t)\, dt, \tag{2.10}$$

and then

$$\psi_1 = -\int_0^x \begin{pmatrix} q_1(t)\sin p - q_2(t)\cos p & q_1(t)\cos p + q_2(t)\sin p \\ q_1(t)\cos p + q_2(t)\sin p & -q_1(t)\sin p + q_2(t)\cos p \end{pmatrix} dt,$$

$$p = z(2t - x), \tag{2.11}$$

$$\psi_2 = \int_0^x e^{-zJ(x-t)}\, JV(t)\, \psi_1(t, z, q)\, dt$$

$$= \int_0^x dt_1 \int_0^{t_1} e^{-zJ(x-2t_1+2t_2)}\, V(t_1)\, V(t_2)\, dt_2. \tag{2.12}$$

Proceeding by induction,

$$\psi_{2n}(x, z, q) = \int_0^x dt_1 \cdots \int_0^{t_{2n}} e^{-zJ(x-t_1+2t_2 \cdots +2t_{2n})}\, V(t_1) \cdots V(t_{2n})\, dt_{2n}, \tag{2.13}$$

$$\psi_{2n+1}(x, z, q) = \int_0^x dt_1 \cdots \int_0^{t_{2n+1}} e^{-zJ(x-t_1+2t_2 \cdots -2t_{2n+1})} \times JV(t_1) \cdots V(t_{2n+1})\, dt_{2n+1}. \tag{2.14}$$

Introduce the Banach spaces $\ell^p = \{f = \{f_n\}_{\mathbb{Z}}, \|f\|^p_{(p)} \equiv \sum |f_n|^p < \infty\}, p \geqslant 1$. In analogy to the notation $O(1/n)$ we use the notation $l^d(n)$, $d \geqslant 1$, for an arbitrary sequence of numbers, which is an element of l^d (see [PT]). For instance,

$$a_n = b_n + l^d(n) \quad \text{is equivalent to} \quad a_n = b_n + c_n, \qquad \sum |c_n|^d < \infty.$$

Now we will prove the basic results about the functions ψ. Let $M(z, q) = \psi(1, z, q)$ and define the functions $(x, y)_n \equiv (J_n x, y)$, $x, y \in \mathbb{R}^2$, $n = 1, 2$. We have

LEMMA 2.1. (i) *For each $q \in H_C$ and $z \in \mathbb{C}$ there exists a unique solution ψ of Eq.* (2.6) *which has the form* (2.7)–(2.9) *and series* (2.7) *converge uniformly and absolutely on bounded subsets of* $[0, 1] \times \mathbb{C} \times H_C$. *For each $x \in [0, 1]$ the function $\psi(x, z, q)$ is entire on $\mathbb{C} \times H_C$. Moreover, ψ is analytic as a map from $\mathbb{C} \times H_C$ into $W^2_{1C}(0, 1)$ and the following estimate is fulfilled*:

$$|\psi(x, z, q)| \leqslant e^{|\mathrm{Im}\, z|\, x + \|q\|_C}. \tag{2.15}$$

If the sequence q^ν converges weakly to q in H_C, as $\nu \to \infty$, then $\psi(x, z, q^\nu) \to \psi(x, z, q)$ uniformly on bounded subsets of $[0, 1] \times \mathbb{C}$.

(ii) *The derivatives of* $M(z, q)$ *with respect to* q *have the forms*

$$(\partial_1 M(z,q))(t) = \begin{pmatrix} \tilde{\vartheta}_1(\vartheta,\varphi)_1 - \tilde{\varphi}_1(\vartheta,\vartheta)_1 & \tilde{\vartheta}_1(\varphi,\varphi)_1 - \tilde{\varphi}_1(\vartheta,\varphi)_1 \\ \tilde{\vartheta}_2(\vartheta,\varphi)_1 - \tilde{\varphi}_2(\vartheta,\vartheta)_1 & \tilde{\vartheta}_2(\varphi,\varphi)_1 - \tilde{\varphi}_2(\vartheta,\varphi)_1 \end{pmatrix}, \tag{2.16}$$

$$(\partial_2 M(z,q))(t) = \begin{pmatrix} \tilde{\vartheta}_1(\vartheta,\varphi)_2 - \tilde{\varphi}_1(\vartheta,\vartheta)_2 & \tilde{\vartheta}_1(\varphi,\varphi)_2 - \tilde{\varphi}_1(\vartheta,\varphi)_2 \\ \tilde{\vartheta}_2(\vartheta,\varphi)_2 - \tilde{\varphi}_2(\vartheta,\vartheta)_2 & \tilde{\vartheta}_2(\varphi,\varphi)_2 - \tilde{\varphi}_2(\vartheta,\varphi)_2 \end{pmatrix}, \tag{2.17}$$

where $\vartheta = \vartheta(t, z, q)$, $\varphi = \varphi(t, z, q)$, *and* $\tilde{\vartheta} = \vartheta(1, z, q)$, $\tilde{\varphi} = \varphi(1, z, q)$.

(iii) *Let* $q, q' \in H$. *Then*

$$|\psi(x,z,q) - \psi_0(x,z,q)| \leqslant \frac{\|q\| + \|q'\|}{|z|} e^{|\mathrm{Im}\, z|\, x + \|q\|}. \tag{2.18}$$

(iv) *Moreover, for any fixed* $d > 1/2$ *the following asymptotic estimates are fulfilled*

$$\begin{aligned} \psi(x,z,q) &= \sum_0^3 \psi_m(x,z,q) + [\ell^d(n)]^4, \\ \psi_m(x,z,q) &= [\ell^d(n)]^m, \qquad m = 1, 2, 3, \end{aligned} \tag{2.19}$$

as $n \to \infty$, *uniformly on* $[0,1] \times \{|z - \pi n| \leqslant \pi\} \times B_C(p, \varepsilon_p)$ *for each fixed* $p \in H_C$.

Remark. $[\ell^d(n)]^2$ in (2.19) means that there exist two sequences $\beta = \{\beta_n(q)\}, \alpha = \{\alpha_n(q)\} \in \ell^2$ such that $\|\beta\|_{(d)} \leqslant 1$, $\|\alpha\|_{(d)} \leqslant 1$, and $\psi_2(x,z,q) = [\ell^d(n)]^2 = \alpha_n(q)\, \beta_n(q)\, O(1)$, as $n \to \infty$, uniformly on bounded subsets of $[0,1] \times \{|z - \pi n| \leqslant \pi\} \times H_C$.

We consider the following 2×2-matrix valued fundamental solution $\Psi = \Psi(x, z, q, t)$ with the parameter t:

$$J\Psi' + V(x+t)\,\Psi = z\Psi, \qquad \Psi(0,z,q,t) = I,\ t \in \mathbb{R},\ z \in \mathbb{C},\ q \in H_C. \tag{2.20}$$

We need the simple properties of Ψ:

$$\Psi(x-t, z, q, t) = \psi(x,z,q)\, \psi(t,z,q)^{-1}, \tag{2.21}$$

$$\psi(x+1, z, q) = \psi(x,z,q)\, \psi(1,z,q). \tag{2.22}$$

We need the following identities from [K9],

$$\begin{aligned} \Psi(1,z,q,t) &= \begin{pmatrix} \vartheta_1(1,z,q,t) & \varphi_1(1,z,q,t) \\ \vartheta_2(1,z,q,t) & \varphi_2(1,z,q,t) \end{pmatrix} \\ &= \begin{pmatrix} \varphi_2(\vartheta_1\tilde{\vartheta}_1 + \varphi_1\tilde{\vartheta}_2) - \vartheta_2(\vartheta_1\tilde{\varphi}_1 + \varphi_1\tilde{\varphi}_2) & -\varphi_1(\vartheta_1\tilde{\vartheta}_1 + \varphi_1\tilde{\vartheta}_2) + \vartheta_1(\vartheta_1\tilde{\varphi}_1 + \varphi_1\tilde{\varphi}_2) \\ \varphi_2(\vartheta_2\tilde{\vartheta}_1 + \varphi_2\tilde{\vartheta}_2) - \vartheta_2(\vartheta_2\tilde{\varphi}_1 + \varphi_2\tilde{\varphi}_2) & -\varphi_1(\vartheta_2\tilde{\vartheta}_1 + \varphi_2\tilde{\vartheta}_2) + \vartheta_1(\vartheta_2\tilde{\varphi}_1 + \varphi_2\tilde{\varphi}_2) \end{pmatrix}, \end{aligned} \tag{2.23}$$

where $\vartheta=\vartheta(t, z, q)$, $\varphi=\varphi(t, z, q)$, and $\tilde{\vartheta}=\vartheta(1, z, q)$, $\tilde{\varphi}=\varphi(1, z, q)$. Define the function $\Delta_1(z, q)=(1/2) \operatorname{Tr} \psi_2(1, z, q)$ and rewrite one in the needed form (see [K9])

$$\Delta_1(z, q)=\int_0^1 dx \int_0^x [(q_1(x)\, q_1(t)+q_2(x)\, q_2(t)) \cos p \\ +(q_1(x)\, q_2(t)-q_1(t)\, q_2(x)) \sin p]\, dt, \tag{2.24}$$

where $p=z(1-2x+2t)$, and

$$\Delta_1(\pi n, q)=\frac{(-1)^n}{2}\, \hat{q}_n^2. \tag{2.25}$$

Below we need the following result [K9].

LEMMA 2.2. (i) *The functions $\Delta(\cdot, \cdot)$ is entire on $\mathbb{C}\times H_C$ and has the form*

$$\Delta(z, q)=\cos z+\Delta_1(z, q)+\Delta_2(z, q), \qquad \Delta_2(z, q)=\tfrac{1}{2} \sum_{n\geqslant 2} \operatorname{Tr} \psi_{2n}(1, z, q), \tag{2.26}$$

where series converge uniformly and absolutely on bounded subsets of $\mathbb{C}\times H_C$ and the following estimate is fulfilled:

$$|\Delta(z, q)|\leqslant e^{|\operatorname{Im} z|\, x+\|q\|}. \tag{2.27}$$

If the sequence q^ν converges weakly to q in H_C, as $\nu\to\infty$, then $\Delta(z, q^\nu)\to\Delta(z, q)$ uniformly on bounded subsets of $\mathbb{C}$.

(ii) *Their derivatives with respect to q have the forms*

$$(\partial_1 \Delta)(t, z, q)=-\tfrac{1}{2}\,(\vartheta_2(1, z, q, t)+\varphi_1(1, z, q, t)), \tag{2.28}$$

$$(\partial_2 \Delta)(t, z, q)=\tfrac{1}{2}\,(\vartheta_1(1, z, q, t)-\varphi_2(1, z, q, t)). \tag{2.29}$$

In particular, if $|\gamma_n|=0$, then $\partial\Delta(z, q)=0$ at $z=m_n(q)$.

(iii) *For fixed $z\in\mathbb{C}$ the function $\Delta(z, q)$ is even with respect to $q\in H_C$ and*

$$\Delta(z, -q)=\Delta(z, q), \qquad q\in H_C. \tag{2.30}$$

Below we need the asymptotics estimates of the Lyapunov function from [K9]. Let $q\in H_C$. Then the following asymptotic estimates, for any fixed $d>1$ are fulfilled;

$$\Delta_1(z, q) = \ell^d(n), \qquad \Delta_2(z, q) = (\ell^d(n))^2, \tag{2.31}$$

$$(\partial_1 \Delta(z, q))(t) = \int_0^1 (q_1(x+t) \cos z(2x-1) + q_2(x+t) \sin z(2x-1)) \times dx + \ell^d(n), \tag{2.32}$$

$$(\partial_2 \Delta(z, q))(t) = \int_0^1 (q_1(x+t) \sin z(2x-1) - q_2(x+t) \cos z(2x-1)) \times dx + \ell^d(n), \tag{2.33}$$

as $|z - \pi n| \leqslant \pi$, and

$$\left(\frac{\partial^m}{\partial z^m} \partial \Delta(z, q)\right)(t) = \ell^2(n), \qquad \text{at} \quad z = \pi n, m \geqslant 1, \tag{2.34}$$

$$(\partial \Delta(z, q))(t) = d_q \hat{q}_n^2 + \ell^d(n), \qquad \text{if} \quad z - \pi n = \ell^2(n), \tag{2.35}$$

as $|z| \to \infty$. These estimates are infinitely differentiable with respect to z and are satisfied uniformly on bounded subsets $[0, 1] \times H_C$.

3. PROPERTIES OF FUNCTIONS $z_n(\cdot)$ AND $|h_n(\cdot)|^2$

For $q, \varepsilon > 0$ we define the new function $\tilde{q}$ and the constants N_q, ε_q by the formulas

$$\tilde{q}(x) = \sum_{-M}^{M} e^{2\pi n x J} \hat{q}_n, \qquad N_q = 24(\|q\| + \|\tilde{q}'\|)\, e^{2\|q\|}, \qquad \varepsilon_q = \frac{1}{4^4} e^{-3\|q\|}, \tag{3.1}$$

where M is defined by the condition $\|q - \tilde{q}\| \leqslant \varepsilon$. Introduce the contour $c_n(r) = \{z\colon |z - \pi n| = r\}, r > 0$. In order to study our mapping, we have to consider the function $z_n(q)$, recall that $\dot{\Delta}(z_n(q), q) = 0$.

LEMMA 3.1. (i) *Let $q \in H_C$ and $\varepsilon_q = 4^{-4} \exp(-3\|q\|)$. Then for any integer $N \geqslant N_q$ and for each $p \in B_C(q, \varepsilon_q)$ the function $\dot{\Delta}(z, p)$ has exactly $2N+1$ roots, counted with multiplicities, in the disc $\{z\colon |z| < \pi(N + (1/2))\}$ and for each $|n| > N$, it has exactly one simple root in the domain $\{z\colon |z - \pi n| < 1\}$. There are no other roots.*

(ii) *Each function $z_n(\cdot)$, $n \in \mathbb{Z}$, is compact and real analytic on H. Its gradient is given by the formula*

$$(d_q z_n(q))(x) = -\frac{(\partial \dot{\Delta})(x, z_n(q), q)}{\ddot{\Delta}(z_n(q), q)}. \tag{3.2}$$

Moreover, for any fixed $d>1$ the following asymptotic estimates are fulfilled,

$$z_n(q) = \pi n + \ell^d(n), \tag{3.3}$$

$$(d_q z_n(q))(x) = (-1)^n \partial \dot{\Delta}(\pi n, q) + \ell^d(n) = \ell^2(n), \tag{3.4}$$

as $|n| \to \infty$, uniformly on $[0,1] \times B_C(p, \varepsilon_p)$ for each fixed $p \in H$.

Proof. (i) Fix N and let $N_1 > N$ be another integer. Let z belong to the contours $|z| = \pi(N+(1/2))$, $|z| = \pi(N_1+(1/2))$, $c_n(1)$, $|n| > N$. Recall that for q we defined the functions $\tilde{q}$ and the integer $M \geqslant 1$ (see (3.1)) such that $\|q - \tilde{q}\| \leqslant \varepsilon$. Using the estimates (2.28) we get

$$|\Delta(z, \tilde{q}) - \cos z| \leqslant \frac{\|q\| + \|\tilde{q}'\|}{|z|} e^{\|q\| + |\mathrm{Im}\, z|},$$

and substituting it in the identity

$$\dot{\Delta}(z, \tilde{q}) + \sin z = \frac{1}{2\pi i} \int_{|z-t|=1} \frac{(\Delta(t, \tilde{q}) - \cos t)\, dt}{(t-z)^2},$$

we obtain

$$\begin{aligned} |\dot{\Delta}(z, \tilde{q}) + \sin z| &\leqslant \frac{\|q\| + \|\tilde{q}'\|}{2\pi} \int_{|z-t|=1} \frac{e^{\|q\| + |\mathrm{Im}\, t|}\, |dt|}{|t|} \\ &\leqslant 2 \frac{(\|q\| + \|\tilde{q}'\|)}{|z|} e^{1 + \|q\| + |\mathrm{Im}\, z|}. \end{aligned} \tag{3.5}$$

We have the identities

$$\dot{\Delta}(z, p) - \dot{\Delta}(z, \tilde{q}) = \frac{1}{2\pi i} \int_{|z-t|=1} \frac{(\Delta(t, p) - \Delta(t, \tilde{q}))}{(t-z)^2}\, dt, \tag{3.6}$$

$$\Delta(t, p) - \Delta(t, \tilde{q}) = \int_0^1 (\partial \Delta(t, \tilde{q} + sv), v)\, ds, \qquad v = p - \tilde{q}, \tag{3.7}$$

where $(\cdot, \cdot)$ is the scalar product in H_C. Identities (2.28–29) and estimate (2.15) yield

$$|\partial\, \Delta(z, \tilde{q} + sv, t)| \leqslant e^{|\mathrm{Im}\, z| + \|\tilde{q} + sv\|} \leqslant e^{|\mathrm{Im}\, z| + \|q\| + 2\varepsilon}, \qquad \varepsilon = \varepsilon_q, \tag{3.8}$$

since $\|v\| \leqslant \|p - q\| + \|q - \tilde{q}\| \leqslant 2\varepsilon$. Substituting this estimate into (3.7) and using (3.6) we have

$$|\dot{\Delta}(z, p) - \dot{\Delta}(z, \tilde{q})| \leqslant 2\varepsilon e^{|\mathrm{Im}\, z| + \|q\| + 2\varepsilon}. \tag{3.9}$$

Using (3.5) and (3.9) we deduce that

$$|\dot{\Delta}(z,p)+\sin z| \leqslant |\dot{\Delta}(z,p)-\dot{\Delta}(z,\tilde{q})| + |\dot{\Delta}(z,\tilde{q})+\sin z| \leqslant \tfrac{1}{4}\, e^{|\mathrm{Im}\, z|}\, C,$$

$$C = 8e^{\|q\|}\left(\varepsilon e^{2\varepsilon} + \frac{\|q\| + \|\tilde{q}'\|}{|z|}\, e\right).$$

Therefore, the simple estimate $\exp|\mathrm{Im}\, z| < 4|\sin z|$ on all contours (see [PT]) yields

$$|\dot{\Delta}(z,p)+\sin z| \leqslant C\,|\sin z|. \tag{3.10}$$

By assumptions, $C<1$ on all contours. Hence, by Rouché's theorem, $\dot{\Delta}(\cdot,p)$ has as many roots, counted with multiplicities, as $\sin z$ in each of the bounded domains and the remaining unbounded region. Since $\sin z$ has only simple roots πn, $n\in\mathbb{Z}$, and since $N_1>N$ can be chosen arbitrarily large, the point (i) of Lemma 3.1 follows.

(ii) In order to prove compactness of $z_n(q)$, suppose the sequence q^{ν}, $\nu\geqslant 1$, converges weakly to q in H. Let $z_n^0(q)$, $n\in\mathbb{Z}$, be the zeros of the function $\Delta(z,q)$, $z\in\mathbb{R}$. Define the segments $I_n^0=[z_n^0(q), z_{n+1}^0(q)]$, $n\in\mathbb{Z}$. In each I_n^0, $n\in\mathbb{Z}$, there exists exactly one zero $z_n(q)$ of the function $\dot{\Delta}(z,q)$, and

$$z_n^0(q)<z_n(q)<z_{n+1}^0(q), \qquad n\in\mathbb{Z}, \tag{3.11}$$

since the function $\Delta(\cdot,q)$ is entire with real zeros (see [Kr]). Using Lemma 2.2 we obtain $\Delta(z,q^{\nu})\to\Delta(z,q)$ and $\dot{\Delta}(z,q^{\nu})\to\dot{\Delta}(z,q)$ as $\nu\to\infty$, uniformly on bounded subsets of $\mathbb{C}$. Then by (3.11), in each I_n^0, $|n|\leqslant N$, for any fixed $N\geqslant 2$, there exists exactly one zero $z_n(q^{\nu})$ of the function $\dot{\Delta}(z,q^{\nu})$ for large ν. For small $\delta>0$ we introduce the intervals

$$I_n=[z_n(q)-\delta, z_n(q)+\delta]\subset I_n^0, \qquad |n|\leqslant N.$$

If δ is sufficiently small, then these intervals are all disjoint and $I_n\subset I_n^0$, $|n|\leqslant N$. The function $\dot{\Delta}(z,q)$ changes sign on each of them, since $z_n(q)$ is a simple root.

As $\nu\to\infty$, the functions $\dot{\Delta}(z,q^{\nu})$ converge to $\dot{\Delta}(z,q)$ uniformly on $\bigcup I_n$ by Lemma 2.2. Hence, for sufficiently large ν, they also change sign on each I_n, so they must all have one root in this interval, since the functions $\dot{\Delta}(z,q^{\nu})$ have exactly one zero $z_n(q^{\nu})$ on each I_n^0, and $I_n\subset I_n^0$, $|n|\leqslant N$. This yields $|z_n(q^{\nu})-z_n(q)|<\delta$, $|n|\leqslant N$, for all sufficiently large ν. It follows that $z_n(q^{\nu})\to z_n(q)$, as $\nu\to\infty$, since N and $\varepsilon>0$ were arbitrary. Thus, $z_n(q)$ is a compact functions of q.

To prove real analyticity, we fix $r\in H$. Then $\dot{\Delta}(z_n(r),r)=0$ and $\ddot{\Delta}(z_n(r),r)\neq 0$. Now, the implicit function theorem guarantees the existence of a function $\tilde{z}_n$ defined on some small neighborhood $W \subset H$ of r and such that

$$\dot{\Delta}(\tilde{z}_n(q), q) = 0, \qquad \tilde{z}_n(r) = z_n(r)$$

on W. Furthermore, $\tilde{z}_n(q)$ is real analytic. On the other hand, $z_n(q)$ is also a continuous function on W satisfying $\dot{\Delta}(z_n(q), q) = 0$. Therefore, by uniqueness, $\tilde{z}_n(q) = z_n(q)$ on W, and so $z_n(q)$ is real analytic.

To calculate the gradient, we observe that $\dot{\Delta}(z_n(q), q) = 0$. Hence

$$0 = d_q\{\dot{\Delta}(z_n(q), q)\} = \ddot{\Delta}(z_n(q), q)\, d_q z_n + \partial\dot{\Delta}(z, q),$$

which implies (3.2).

Using (i), we obtain $|z_n(q) - \pi n| \leqslant 1$, $|n| \to \infty$. We can improve (i). The asymptotic estimates (2.31) yield $\dot{\Delta}(z_n(q), q) = -\sin z_n(q) + \ell^d(n)$. Then $\sin(z_n(q) - \pi n) = \ell^d(n)$ which implies (3.3). Hence from (2.34), (2.31), (3.3) we conclude

$$\begin{aligned} d_q z_n(q) &= -\frac{\partial\dot{\Delta}(z_n(q), q)}{\ddot{\Delta}(z_n(q), q)} = -\frac{\partial\dot{\Delta}(\pi n, q) + \ell^d(n)}{\cos z_n(q) + \ell^d(n)} \\ &= (-1)^n\, \partial\dot{\Delta}(\pi n, q) + \ell^d(n) = \ell^2(n). \quad \blacksquare \end{aligned}$$

We need the following results concerning the function $r_n = h_n^2$.

Lemma 3.2. *Each function $r_n(\cdot)$, $n \in \mathbb{Z}$, is compact and real analytic on H. Its gradient is given by the formula*

$$(d_q r_n(q))(x) = (-1)^n \frac{(\partial\, \Delta(z_n(q), q))(x)}{(d \cosh \sqrt{r_n}/dr_n)}, \qquad n \in \mathbb{Z}. \tag{3.12}$$

Moreover, for any fixed $d > 1$ the following asymptotics are fulfilled;

$$r_n(q) = \hat{q}_n^2 + (\ell^d(n))^3, \tag{3.13}$$

$$(d_q r_n(q))(x) = d_q \hat{q}_n^2 + \ell^d(n), \tag{3.14}$$

as $|n| \to \infty$, uniformly on $[0, 1] \times B_C(p, \varepsilon_p)$ for each fixed $p \in H$.

Proof. Since $\Delta(z_n(q), q)$ is compact, so is $r_n(q)$. In order to prove real analyticity, we fix $v \in H$. Then $(-1)^n\, \Delta(z_n(v), v) = \cosh \sqrt{r_n}$ and $(\cosh \sqrt{r_n})'_{r_n} \neq 0$. Now, the implicit function theorem guarantees the existence of a unique continuous function $\tilde{r}_n$ defined on some small neighborhood $W \subset H$ of v and such that

$$(-1)^n\, \Delta(z_n(q), q) = \cosh \sqrt{\tilde{r}_n(q)}, \qquad \tilde{r}_n(v) = r_n(v),$$

on W. Furthermore, $\tilde{r}_n(q)$ is real analytic. On the other hand, $r_n(q)$ is also a continuous function on W satisfying $(-1)^n \Delta(z_n(q), q) = \cosh\sqrt{r_n(q)}$. Therefore, by uniqueness, $\tilde{r}_n(q) = r_n(q)$ on W, and so $r_n(q)$ is real analytic.

To calculate the gradient, we observe that $(-1)^n \Delta(z_n(q), q) = \cosh \sqrt{r_n(q)}$. Hence the implicit function theorem yields

$$(-1)^n (d\cosh\sqrt{r_n}/dr_n)\, d_q r_n = \partial\, \Delta(z_n(q), q) + \dot{\Delta}(z_n(q), q)\, d_q z_n(q),$$

and the identity $\dot{\Delta}(z_n(q), q) = 0$ implies (3.12).

We need a simple estimate concerning the analytic function $w = w(z) = \cosh\sqrt{z} - 1$, where $z \in \mathbb{C}$. There exists a domain $B \subset \mathbb{C}$, $0 \in B$, such that the new analytic function $w_1(z) \equiv w(z)$, $z \in B$, has an inverse function $g(w)$, analytic in $|w| < 2$, and the following asymptotics are fulfilled:

$$z = g(w) = 2w(1 - w/6 + \cdots), \qquad w \to 0. \tag{3.15}$$

Now we let $n \to \infty$. The function $t_n(q) \equiv (-1)^n \Delta(z_n(q), q) - 1$ is real analytic and (3.3), (2.25), (2.31) imply the asymptotics

$$t_n = [(-1)^n \cos z_n + \hat{q}_n^2/2 + (\ell^d(n))^2] - 1 = \frac{\hat{q}_n^2}{2} + (\ell^d(n))^2 \tag{3.16}$$

uniformly on $B_C(p, \varepsilon_p)$. Hence the function $t_n(r_n) = \cosh\sqrt{r_n} - 1$ has an inverse $r_n = g(t_n)$ for $|t_n| < 2$, and (3.15)–(3.16) yield

$$r_n = 2t_n(1 - t_n/6 + \cdots) = \hat{q}_n^2 + (\ell^d(n))^2.$$

Substituting the asymptotic estimates (3.3), (3.13), (2.35) into (3.12) we obtain

$$d_q r_n = (-1)^n \frac{\partial\, \Delta(z_n(q), q)}{(\sin\mathrm{h}\sqrt{r_n}/2\sqrt{r_n})} = \frac{d_q \hat{q}_n^2 + \ell^d(n)}{1 + O(r_n)} = d_q \hat{q}_n^2 + \ell^d(n),$$

as $n \to \infty$, uniformly on $[0, 1] \times B_C(p, \varepsilon_p)$. ∎

Below we need the following result from [K9].

LEMMA 3.3. (i) *Let $q \in H_C$ and $\varepsilon_q = 4^{-4} \exp(-3\|q\|)$. Then for each integer $N > N_q$ and any $p \in B_C(q, \varepsilon_q)$ the function $\varphi_1(1, z, p)$ has exactly $2N+1$ roots, counted with multiplicities, in the disc $\{z\colon |z| < \pi(N + (1/2))\}$ and for each $|n| > N$, exactly one simple root in the domain $\{z\colon |z - \pi n| < 1\}$. There are no other roots.*

(ii) *Each function* $m_n(\cdot)$, $n \in \mathbb{Z}$, *is compact and real analytic on* H. *Its gradient is given by the formula*

$$(d_q m_n(q))(x)$$
$$= -\frac{((\varphi(t, z, q), \varphi(t, z, q))_1, (\varphi(t, z, q), \varphi(t, z, q))_2)}{\dot{\varphi}_1(1, z, q)\, \varphi(1, z, q)}, \qquad z = m_n(q), \tag{3.17}$$

and

$$\|\varphi(\cdot, m_n(q), q)\|^2 = \dot{\varphi}_1(1, m_n(q), q)\, \varphi_2(1, m_n(q), q) > 0. \tag{3.18}$$

Moreover, for any fixed $d > 1$ *the following asymptotic estimates are fulfilled,*

$$m_n(q) = \pi n - q_{1(cn)} - q_{2(sn)} + \ell^d(n), \tag{3.19}$$

$$(d_q m_n(q))(x) = -(\cos 2\pi n x, \sin 2\pi n x) + \ell^2(n), \tag{3.20}$$

as $|n| \to \infty$, *uniformly on* $[0, 1] \times B_C(p, \varepsilon_p)$ *for each fixed* $p \in H$.

4. BASIC PROPERTIES OF FUNCTIONS $h_n(\cdot)$

Recall that $(x, y)_n = (J_n x, y)$, $x, y \in \mathbb{R}^2$, $n = 1, 2$. Define the vectors

$$a_n(x, q) = ((\vartheta(x, m_n(q), q), \varphi(x, m_n(q), q))_1,$$
$$(\vartheta(x, m_n(q), q), \varphi(x, m_n(q), q))_2) \in \mathbb{R}^2, \qquad n \in \mathbb{Z}.$$

We need the following result concerning the mapping $h_{n2}(q) = -\log[(-1)^n \varphi_2(1, m_n(q), q)]$.

LEMMA 4.1. *Each function* $h_{n2}(\cdot)$, $n \in \mathbb{Z}$, *is compact and real analytic on* H. *Its gradient is given by the formulas*

$$d_q h_{n2} = -\frac{1}{\varphi_2(1, m_n, q)} \{\dot{\varphi}_2(1, m_n, q)\, d_q m_n(q) + \partial\varphi_2(1, m_n, q)\}, \tag{4.1}$$

$$d_q h_{n2} = \frac{(-1)^n}{\sinh h_{n2}} \{\dot{\Delta}(m_n(q), q)\, d_q m_n(q) + \partial\Delta(m_n(q), q)\}, \qquad h_{n2} \neq 0, \tag{4.2}$$

$$d_q h_{n2} = a_n(\cdot, q) + \varphi_2(1, m_n(q), q)\, \dot{\vartheta}_1(1, m_n(q), q)\, d_q m_n(q). \tag{4.3}$$

Moreover, for any fixed $d>1$ the following asymptotic estimates are fulfilled,

$$h_{n2}(q) = -q_{1(sn)} + q_{2(cn)} + \ell^d(n), \tag{4.4}$$

$$(d_q h_{n2}(q))(x) = (-\sin 2\pi nx, \cos 2\pi nx) + \ell^2(n), \tag{4.5}$$

as $n \to \pm\infty$, uniformly on subsets of $[0, 1] \times B_C(p, \varepsilon_p)$ for each fixed $p \in H$.

Proof. $\varphi_2(1, z, q)$ and $m_n(q)$ are compact, real analytic functions of (z, q) and q, respectively. Then $h_{n2}(q) = -\log[(-1)^n \varphi_2(1, m_n(q), q)]$ is also compact and real analytic, since $\varphi_2(1, m_n(q), q)$ never vanishes. The differentiation rule yields

$$d_q h_{n2}(q) = -\frac{1}{\varphi_2(1, z, q)} [\dot{\varphi}_2(1, z, q)\, d_q\, m_n(q) + \partial\varphi_2(1, z, q)], \qquad \text{at}$$

$$z = m_n(q).$$

Using (1.4), Lemmas 2.2, 3.3, and the implicit function theorem we deduce that

$$(\sinh h_{n2})\, d_q h_{n2} = (-1)^n [\partial\Delta(m_n(q), q) + \dot{\Delta}(m_n(q), q)\, d_q m_n(q)], \qquad h_{n2} \neq 0. \tag{4.6}$$

Identities (2.16)–(2.17) imply

$$\begin{aligned}\partial\varphi_2(1, m_n(q), q) &= \vartheta_2(1, m_n(q), q)((\varphi, \varphi)_1, (\varphi, \varphi)_2)\\ &\quad - \varphi_2(1, m_n(q), q)((\vartheta, \varphi)_1, (\vartheta, \varphi)_2),\end{aligned} \tag{4.7}$$

where $\varphi = \varphi(t, z, q)$, $\vartheta = \vartheta(t, z, q)$. Substituting the last identity into (4.1), and using (3.17), $\vartheta_1(1, m_n(q), q)\, \varphi_2(1, m_n(q), q) = 1$, and $\vartheta_1\varphi_2 - \vartheta_2\varphi_1 = 1$, we obtain

$$\begin{aligned} d_q h_{n2} &= a_n - \frac{\dot{\varphi}_2(1, m_n(q), q)}{\varphi_2(1, m_n(q), q)} d_q m_n(q) - \frac{\vartheta_2(1, m_n(q), q)}{\varphi_2(1, m_n(q), q)} ((\varphi, \varphi)_1, (\varphi, \varphi)_2)\\ &= a_n - [\dot{\varphi}_1(1, m_n(q), q)\, \vartheta_2(1, m_n(q), q)\\ &\quad - \dot{\varphi}_2(1, m_n(q), q)\, \vartheta_1(1, m_n(q), q)]\, d_q m_n(q), \end{aligned}$$

and we have (4.3). Lemma 2.1 and (3.19) yield

$$
\begin{aligned}
e^{-h_{n2}} &= (-1)^n\, \varphi_2(1, m_n(q), q) \\
&= (-1)^n \bigg(\cos m_n(q) - \int_0^1 [-q_1(x) \sin m_n(q)(2x-1) \\
&\qquad + q_2(x) \cos m_n(q)(2x-1)]\, dx \bigg) + \ell^d(n)) \\
&= \cos(m_n(q) - \pi n) + q_{1(sn)} - q_{2(cn)} + \ell^d(n) \\
&= 1 + q_{1(sn)} - q_{2(cn)} + \ell^d(n),
\end{aligned}
$$

and then

$$e^{-h_{n2}} = (-1)^n\, \varphi_2(1, m_n(q), q) = 1 + q_{1(sn)} - q_{2(cn)} + \ell^d(n), \tag{4.8}$$

which gives (4.4). Using (4.7)–(4.8) and Lemma 2.1, we get

$$\vartheta_2(1, m_n(q), q)((\varphi, \varphi)_1, (\varphi, \varphi)_2) = \ell^2(n), \tag{4.9}$$

$$
\begin{aligned}
(\vartheta(x, z, q), \varphi(x, z, q))_1 &= \vartheta_1 \varphi_1 - \vartheta_2 \varphi_2 \\
&= -2 \cos y \sin y + \ell^2(n) = -\sin 2y + \ell^2(n)
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
(\vartheta(x, z, q), \varphi(x, z, q))_2 &= \vartheta_1 \varphi_2 + \vartheta_2 \varphi_1 \\
&= \cos^2 y - \sin^2 y + \ell^2(n) = \cos 2y + \ell^2(n),
\end{aligned} \tag{4.11}
$$

where $y = 2\pi n x$. Relations (2.19), (3.19) imply

$$\dot\varphi_2(1, m_n, q) = -\sin m_n + \ell^2(n) = \ell^2(n). \tag{4.12}$$

Substituting (4.7)–(4.12) into (4.1) we obtain (4.5). ▮

Now we consider the function $h_{n1} = ||h_n|^2 - h_{n2}^2|^{1/2} \operatorname{sign}(z_n - m_n)$.

LEMMA 4.2. *Each function $h_{n1}(\cdot)$, $n \in \mathbb{Z}$, is compact and real analytic on H. There exists a compact, real analytic and positive function $b_n(\cdot)$ on H such that*

$$h_{n1}(q) = (z_n(q) - m_n(q))\, b_n(q), \qquad q \in H, \tag{4.13}$$

and if $h_{n1}(v) = 0$ for some $v \in H$, then

$$d_q h_{n1}(v) = b_n(v)(d_q z_n(v) - d_q m_n(v)). \tag{4.14}$$

Moreover, for any fixed $d > 1$, *the following asymptotics are fulfilled,*

$$b_n(q) = 1 + \ell^2(n), \qquad d_q b_n(q) = \ell^2(n), \tag{4.15}$$

$$h_{n1}(q) = q_{1(cn)} + q_{2(sn)} + \ell^d(n), \tag{4.16}$$

$$(d_q h_{n1}(q))(x) = (\cos 2\pi nx, \sin 2\pi nx) + \ell^2(n), \tag{4.17}$$

as $n \to \infty$, *uniformly on* $[0, 1] \times B_C(p, \varepsilon_p)$ *for each fixed* $p \in H$.

Proof. Introduce the functions $r_{n2} = h_{n2}^2$ and $f_n(q) = f(r_n(q), r_{n2}(q))$, $q \in H$, where

$$f(x, y) = 2\left[\frac{1}{2} + \frac{1}{4!}(x + y) + \frac{1}{6!}(x^2 + xy + y^2) + \cdots\right].$$

Here f is an entire function of two parameters x, y and positive if $x \geqslant 0$, $y \geqslant 0$. Then $f_n(\cdot)$ is compact, real analytic, and positive on H. Using (4.4)–(4.5) and (3.13)–(3.14) we obtain

$$f_n(q) = 1 + \ell^1(n), \qquad d_q f_n(q) = \ell^2(n), \tag{4.18}$$

as $|n| \to \infty$, uniformly on $[0, 1] \times B_C(p, \varepsilon_p)$ for each fixed $p \in H$. Fixing $q \in H$, we apply Taylor's formula for $\Delta(z) = \Delta(z, q)$, $m_n = m_n(q)$, $z_n = z_n(q)$, $\zeta_n = m_n - z_n$, with remainder in integral form, at $z = z_n$:

$$(-1)^n (\Delta(z_n) - \Delta(m_n)) = \zeta_n^2 F_n(q)/2,$$

$$F_n(q) \equiv (-1)^{n+1}\left[\ddot{\Delta}(z_n) + \zeta_n \int_0^1 (1-t)^2 \dddot{\Delta}(z_n + t\zeta_n)\, dt\right].$$

Using the properties of Δ, m_n, z_n we get that the function F_n is compact, real analytic and positive on H. From (2.19)–(2.20), (3.3)–(3.4), (2.30)–(2.31) we deduce that

$$F_n(q) = 1 + \ell^2(n), \qquad d_q F_n(q) = \ell^2(n), \qquad \text{as} \quad |n| \to \infty, \tag{4.19}$$

uniformly on $[0, 1] \times B_C(p, \varepsilon_p)$. Introduce the compact, real analytic and positive function $y_n = F_n(q)/f_n(q)$ on H. Then we can define a compact, real analytic and positive function $b_n = \sqrt{y_n}$ on H. Using (4.18)–(4.19) we obtain the asymptotic estimates (4.15).

Identities (1.3)–(1.4) yield

$$\zeta_n^2 F_n/2 = (-1)^n (\Delta(z_n) - \Delta(m_n)) = \cosh\sqrt{r_n} - \cosh\sqrt{r_{n2}}$$
$$= (r_n - r_{cn})\, f_n(r_n, r_{n2})/2.$$

Therefore, we obtain $r_n - r_{n2} = \zeta_n^2 y_n$, which implies (4.13), indeed

$$h_{n1}(q) = |r_n - r_{n2}|^{1/2} \operatorname{sign}(z_n(q) - m_n(q)) = (z_n(q) - m_n(q))\, b_n(q), \qquad q \in H.$$

Therefore, the function $h_{n1}(\cdot)$ is compact and real analytic on H. Identity (4.13) yields (4.14).

Identity (4.13) and (4.15), (3.19)–(3.20) and (3.3)–(34) imply (4.16)–(4.17). ▮

We need the following result concerning the vectors a_n, $d_q m_n(q)$. Define the symplectic forms

$$(x, y)_0 \equiv (Jx, y) = x_2 y_1 - x_1 y_2, \qquad x, y \in \mathbb{R}^2,$$

$$f \wedge g = \int_0^1 (f(x), g(x))_0 dx, \qquad f, g \in H,$$

and note that $f \wedge f = 0$.

Lemma 4.3. *For any $n, p \in \mathbb{Z}$, the following identities are fulfilled*

$$d_q m_n(q) \wedge d_q m_p(q) = 0, \tag{4.20}$$

$$a_n(q) \wedge d_q m_p(q) = \delta_{np}, \tag{4.21}$$

$$a_n(q) \wedge a_p(q) = 0. \tag{4.22}$$

Proof. We need the simple identity. Let f, g be the solution of the Eq. (1.1) with z, z_1, respectively. Then

$$(f(x), g(x))_0' = (z - z_1)(f(x), g(x)), \qquad x \in [0, 1]. \tag{4.23}$$

Let $\vartheta(t) \equiv \vartheta(t, m_n(q), q)$, $\tilde{\vartheta}(t) \equiv \vartheta(t, m_p(q), q)$. Let $n \neq p$, since the case $n = p$ is simple. Using (3.17) we obtain

$$d_q m_n(q) \wedge d_q m_p(q) = C \int_0^1 v(t)\, dt, \quad v = (\varphi, \varphi)_2\, (\tilde{\varphi}, \tilde{\varphi})_1 - (\varphi, \varphi)_1\, (\tilde{\varphi}, \tilde{\varphi})_2 \quad C \neq 0, \tag{4.24}$$

and

$$\begin{aligned} v &= (2\varphi_1\varphi_2)(\tilde{\varphi}_1^2 - \tilde{\varphi}_2^2) - (\varphi_1^2 - \varphi_2^2)(2\tilde{\varphi}_1\tilde{\varphi}_2) \\ &= 2\varphi_1\tilde{\varphi}_1(\varphi_2\tilde{\varphi}_1 - \varphi_1\tilde{\varphi}_2) + 2\varphi_2\tilde{\varphi}_2(\varphi_2\tilde{\varphi}_1 - \varphi_1\tilde{\varphi}_2) \\ &= 2(\varphi(t), \tilde{\varphi}(t))(\varphi(t), \tilde{\varphi}(t))_0, \end{aligned}$$

and (4.23) implies

$$v(t) = \frac{2(\varphi(t), \tilde{\varphi}(t))'_0 (\varphi(t), \tilde{\varphi}(t))_0}{m_n - m_p}. \tag{4.25}$$

The substitution of (4.25) into (4.24) gives

$$\int_0^1 v(t)\, dt = \frac{(\varphi(t), \tilde{\varphi}(t))_0^2}{m_p - m_n} \Big|_0^1 = 0,$$

and (4.20) is proved. We show (4.22). We have the identity

$$a_n \wedge a_p = \int_0^1 v(t)\, dt,$$

$$v(t) = (\vartheta(t), \varphi(t))_2 \, (\tilde{\vartheta}(t), \tilde{\varphi}(t))_1 - (\vartheta(t), \varphi(t))_1 \, (\tilde{\vartheta}(t), \tilde{\varphi}(t))_2.$$

From (4.23) we obtain

$$\begin{aligned} v &= (\vartheta_1 \varphi_2 + \vartheta_2 \varphi_1)(\tilde{\vartheta}_1 \tilde{\varphi}_1 - \tilde{\vartheta}_2 \tilde{\varphi}_2) + (\vartheta_2 \varphi_2 - \vartheta_1 \varphi_1)(\tilde{\vartheta}_1 \tilde{\varphi}_2 + \tilde{\vartheta}_2 \tilde{\varphi}_1) \\ &= (\vartheta, \tilde{\vartheta})(\varphi, \tilde{\varphi})_0 + (\varphi, \tilde{\varphi})(\vartheta, \tilde{\vartheta})_0 = (m_n - m_p)^{-1} \\ &\quad \times ((\vartheta, \tilde{\vartheta})'_0 \, (\varphi, \tilde{\varphi})_0 + (\varphi, \tilde{\varphi})'_0 \, (\vartheta, \tilde{\vartheta})_0). \end{aligned}$$

The substitution of this identity yields

$$a_n \wedge a_p = \int_0^1 v(t)\, dt = \frac{(\vartheta, \tilde{\vartheta})_0 (\varphi, \tilde{\varphi})_0}{m_n - m_p} \Big|_0^1 = 0.$$

Consider the case of (4.21). Using (3.17) we obtain

$$a_n \wedge d_q m_p = C \int_0^1 v(t)\, dt,$$

$$v(t) = (\vartheta(t), \varphi(t))_2 \, (\tilde{\varphi}(t), \tilde{\varphi}(t))_1 - (\vartheta(t), \varphi(t))_1 \, (\tilde{\varphi}(t), \tilde{\varphi}(t))_2, \quad C \neq 0.$$

and then

$$\begin{aligned} v &= -(\vartheta_1 \varphi_1 - \vartheta_2 \varphi_2)(2\tilde{\varphi}_1 \tilde{\varphi}_2) + (\vartheta_1 \varphi_2 + \vartheta_2 \varphi_1)(\tilde{\varphi}_1^2 - \tilde{\varphi}_2^2) \\ v(t) &= (\vartheta(t), \tilde{\varphi}(t))(\varphi(t), \tilde{\varphi}(t))_0 + (\varphi(t), \tilde{\varphi}(t))(\vartheta(t), \tilde{\varphi}(t))_0. \end{aligned} \tag{4.26}$$

The substitution of (4.23) into the last identity yields

$$v(t)=(m_n-m_p)^{-1}\,((\vartheta(t),\tilde{\varphi}(t))_0'\,(\varphi(t),\tilde{\varphi}(t))_0+(\varphi(t),\tilde{\varphi}(t))_0'\,(\vartheta(t),\tilde{\varphi}(t))_0).$$

Then

$$a_n\wedge d_q m_p=(m_n-m_p)^{-1}\,((\vartheta,\tilde{\varphi})_0\,(\varphi,\tilde{\varphi})_0)|_0^1=0.$$

Now consider the case $n=p$. Then by (4.26),

$$v(t)=(\vartheta(t),\tilde{\varphi}(t))(\varphi(t),\tilde{\varphi}(t))_0=|\varphi(t)|^2,$$

which yields (4.21), as $n=p$. ∎

The gradients of $m_n(q)$, h_{ns} satisfy the following relations and we will use their later on.

LEMMA 4.4.. *For any $n, p\in\mathbb{Z}$, the following identities are fulfilled*

$$d_q m_n(q)\wedge d_q m_p(q)=0, \tag{4.27}$$

$$d_q h_{n2}(q)\wedge d_q m_p(q)=\delta_{np}, \tag{4.28}$$

$$d_q h_{n2}(q)\wedge d_q h_{p2}(q)=0. \tag{4.29}$$

Moreover, the sequence $\{d_q m_n(q), d_q h_{n2}(q),\ n\in\mathbb{Z}\}$, is a basis for H.

Proof. For example, using Lemma 4.3 and (4.3) we obtain

$$d_q h_{n2}(q)\wedge d_q m_p(q)=(a_n-\varphi_2(1,m_n(q),q)\,\dot{\vartheta}_1$$

$$(1,m_n(q),q)\,d_q m_n(q)))\wedge d_q m_p(q)=\delta_{np},$$

The proof of other identities is similar.

Using (3.20), (4.5) for the sequence $\{d_q m_n(q), d_q h_{n2}(q),\ n\in\mathbb{Z}\}$, we obtain

$$(d_q h_{n2}(q))(x)=(-\sin 2\pi nx,\cos 2\pi nx)+\ell^2(n),$$

$$(d_q m_n(q))(x)=-(\cos 2\pi nx,\sin 2\pi nx)+\ell^2(n).$$

The vectors on the right, without the error terms, are an orthonormal basis of H and the error terms are square summable. By (4.27)–(4.29), these vectors are linear independent, since for each vector of them there exist another one, which is perpendicular to all vectors except the given one. Then by the well known result from the functional analysis (see [PT]) the sequence $\{d_q m_n(q), d_q h_{ns}(q)\ n\in\mathbb{Z}\}$, is a basis for H. ∎

5. ANALYTIC ISOMORPHISM

We are now ready to prove the our main theorem.

Proof of Theorem 1.1. We check all conditions of Theorem A for the mapping $h: H \to \ell^2 \oplus \ell^2$.

(i) By Lemma 4.1–4.2, each function $h_n(\cdot), n \in \mathbb{Z}$, is compact and real analytic on H, and the asymptotics (1.8)–(1.9) are fulfilled. The relations (1.8) show that the map $h(q)$ is locally bounded; by the uniform boundedness principle, $h(q)$ is real analytic.

(ii) Using asymptotic estimate (1.10), we see that $d_q h$ is the sum of the operator Φ and a compact operator for all $q \in H$. That is $d_q h - \Phi$ is a compact operator; consequently, $d_q h$ is a Fredholm operator. We prove that the operator $d_q h$ is invertible by contradiction. Let $g \in H$ be a solution of the equation

$$(d_q h(q))\, g = 0, \qquad \text{or} \qquad \{(d_q h_n(q), g) = 0, n \in \mathbb{Z}\},\ g = (g_1, g_2)^T \in H, \quad (5.1)$$

for some fixed $q \in H$. We introduce the function

$$f(z) = \int_0^1 (\partial \Delta)(x, z, q)\, g(x)\, dx, \qquad z \in \mathbb{C}.$$

Now we recall the well known Paley–Wiener result (see [Le]).

Suppose that $F(z)$ is an entire function with

$$\sup_{z \in \mathbb{C}} |F(z)\, e^{-|z|}| < \infty, \qquad \int_{\mathbb{R}} |F(x)|^2\, dx < \infty,$$

and $F(\zeta_n) = 0$ for some sequence $\{\zeta_n\}_{-\infty}^{\infty}$ of distinct real numbers such that $\zeta_n = \pi n + o(1)$ as $|n| \to \infty$. Then $F \equiv 0$.

We will show that f satisfies the conditions above and $\zeta_n = z_n$. The function $r_n = h_{n1}^2 + h_{n2}^2$ is analytic and (5.1) implies $(d_q r_n, g) = 0$. Then (3.12) yields

$$f(z_n) = (-1)^n\, (d \cosh \sqrt{h_n}/dr_n)(d_q r_n, g) = 0.$$

Hence $f(z_n) = 0$ for any $n \in \mathbb{Z}$, and (3.3) implies $z_n = \pi n + o(1)$ as $|n| \to \infty$. Now we have to show that $f \in L^2(\mathbb{R})$. Asymptotic estimates (2.32)–(2.33) yield

$$f(z)=\int_0^1 (\partial \Delta(z,q))(t)\, g(t)\, dt$$

$$= -\tfrac{1}{2}\int_0^1\int_0^1 ([q_1(x+t)\cos z(2t-1)+q_2(x+t)\sin z(2t-1)]\, g_1(t)+$$

$$\times (q_2(x+t)\sin z(2t-1)+q_1(x+t)\cos z(2t-1))\, g_2(t))\, dt\, dy+\ell^d(n)$$

and then

$$f(z)=\int_0^1 [G_1(t)\cos z(2t-1)+G_2(t)\sin z(2t-1)]\, dt+\ell^d(n)),$$

where

$$G_p(y)=\int_0^1 q_p(y+t)(g_1(t)+g_2(t))\, dt,$$

$$p=1,2,\ |z-\pi n|\leqslant \pi,\ |n|\to\infty,\ d>1.$$

So, using (2.27)–(2.29) we see that all conditions for the function F (the Paley–Wiener result) are fulfilled, and we get $f\equiv 0$.

For fixed $q\in H$ we have 3 cases. First, let $h_{n1}=0$. Then by (4.14), $(d_q h_{n1}, g)=b_n(d_q z_n - d_q m_n, g)=0$ and by (3.2) we deduce that $(d_q z_n, g)=0$ and therefore $(d_q m_n, g)=0$

Second, if $h_{n2}\neq 0$, $h_{n1}\neq 0$, then we have by relation (4.2)

$$(-1)^n \sinh h_{n2}(d_q h_{n2}, g)=\dot{\Delta}(m_n(q),q)(d_q m_n, g),$$

and then we have $(d_q m_n, g)=0$, since $f\equiv 0$ and $\Delta(m_n)\neq 0$.

In the last case, let $h_{n2}=0$, $h_{n1}\neq 0$, then by (1.4), $\vartheta_1(1,m_n(q),q)=\varphi_2(1,m_n(q),q)=(-1)^n=\Delta(m_n(q),q)$. Therefore, using the relation (2.16)–(2.17), (3.17) we get

$$\partial\Delta(m_n(q),q)=C d_q m_n(q),$$

for some $C\neq 0$. Then we have

$$f(m_n(q))=(\partial\Delta(m_n(q)), g)=C(d_q m_n(q), g)=0.$$

Hence we obtain $(d_q m_n(q), g)=0$, $(d_q h_{n2}, g)=0$, for any $n\in\mathbb{Z}$. By Lemma 4.4, for any fixed $q\in H$ the vectors $\{(d_q h_{n2}), (d_q m_n), n\in\mathbb{Z}\}$ form a basis of H. Then $g=0$ and the operator $d_q h$ is invertible.

(iii) Estimates (1.7) were proved in [K3].

(iv) By Lemma 4.1–4.2, each mapping $h_n\colon H\to\mathbb{R}^2$, $n\in\mathbb{Z}$, is compact.

(v) We need some results for the operator $T = J\frac{d}{dx} + V(x)$ with a real 1-periodic potential $q, q' \in H$ from [K9]

$$\|q'\| \leqslant \|h\|_1^2 (1 + \|h\|_1^2), \qquad \|h\|_1^2 \equiv \sum (1 + (2\pi n)^{(6.12)}) |h_n|^2. \tag{5.2}$$

Hence, for each $\xi > 0$ the set $\{q : \|h\|_1 < \xi\}$ is compact.

Therefore, all conditions of Theorem A are fulfilled and then h is the real analytic isomorphism between H and $\ell^2 \oplus \ell^2$. (1.8) will be proved in Sect. 6. ∎

6. IDENTITIES AND SIMPLE EXAMPLE OF INVERSE PROBLEM

We consider the Dirac operator T_P acting on the Hilbert space $\mathscr{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and having the form

$$T_P = J\frac{d}{dx} + P(x), \qquad P \equiv \begin{pmatrix} p_1 & p \\ p & p_2 \end{pmatrix}, \qquad \int_0^1 (p_1(t) + p_2(t))\, dt = 0, \tag{6.1}$$

where $p, p_1, p_2 \in L^2(\mathbb{T})$ are real 1-periodic functions of $x \in \mathbb{R}$. Using the standard transformation (see [LS])

$$T_W = \mathscr{U}^* T_P \mathscr{U}, \qquad \text{where } \mathscr{U} = \begin{pmatrix} \cos b(x) & -\sin b(x) \\ \sin b(x) & \cos b(x) \end{pmatrix} = e^{-b(x)J}, \tag{6.2}$$

for some function $b(x)$, and the simple identity $\mathscr{U}J = J\mathscr{U}$, we obtain

$$T_W = J\frac{d}{dx} + W, \qquad \text{where} \quad W = b'I + \mathscr{U}^{-1} P \mathscr{U} = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}. \tag{6.3}$$

Let $c = \cos b$, $s = \sin b$, then we have

$$W = b'I + \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} p_1 & p \\ p & p_2 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$$\times \begin{pmatrix} b' + c^2 p_1 + s^2 p_2 + 2csp & (c^2 - s^2)\, p + sc(p_2 - p_1) \\ (c^2 - s^2)\, p + sc(p_2 - p_1) & b' + s^2 p_1 + c^2 p_2 - 2csp \end{pmatrix}.$$

We take b such that $\operatorname{Tr} W = 0$. Then $w_{11} + w_{22} = 0$ and $2b' + p_1 + p_2 = 0$. Hence the 1-periodic function b has the form

$$b = -\tfrac{1}{2} \int_0^x (p_1(t) + p_2(t))\, dt, \tag{6.4}$$

and

$$W = \begin{pmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{pmatrix}, \tag{6.5}$$

where

$$w_1 = \frac{(p_1 - p_2)}{2} \cos 2b + p \sin 2b, \qquad w_2 = -\frac{(p_1 - p_2)}{2} \sin 2b + p \cos 2b. \tag{6.6}$$

Since $b(x)$ is the 1-periodic function, then we have the Dirac operator with the 1-periodic matrix-function W.

Proof of Theorem 1.2. We consider the Dirac operator T_0 acting on the Hilbert space $\mathscr{H}$ with the form

$$T_0 = J\frac{d}{dx} + P,\ P = \begin{pmatrix} \rho(x)+m & 0 \\ 0 & \rho(x)-m \end{pmatrix}, \qquad \int_0^1 \rho(t)\, dt = 0,$$

where ρ is the potential energy, 1-periodic functions in $x \in \mathbb{R}$, and m is the mass of the particle. We apply the transformation $\mathscr{U}$ from (6.2) to the operator T_0. Identity (6.4) yields

$$b = -\int_0^x \rho(t)\, dt.$$

Then by (6.6),

$$w_1 = m(\cos^2 b - \sin^2 b) = m \cos 2b, \qquad w_2 = 2m \cos b \sin b = m \sin 2b.$$

Therefore

$$W = m\begin{pmatrix} \cos 2b & \sin 2b \\ \sin 2b & -\cos 2b \end{pmatrix},$$

We have $w, w' \in H$. Then there exist $Q_2 = (1/\pi) \int t^2 v(t + i0)\, dt < \infty$ and then the needed identities (see [KK2]) are fulfilled (here $k = u + iv$),

$$\int_0^1 (w_1^2(x) + w_2^2(x))\, dx = \frac{1}{\pi} \iint |z'(k) - 1|^2\, du\, dv = \frac{1}{2} \sum (\mu_n^- z_n^- + \mu_n^+ z_n^+). \tag{6.7}$$

and we have (1.11) since $\int_0^1 (w_1^2(x) + w_2^2(x))\, dx = m^2$. ∎

Proof of (1.8). Consider now the unitary transformation $\Psi = \Psi_{t,r}$ which is defined by

$$\Psi_{t,r} = e^{-bJ}, \qquad b = t - rx, \qquad r = \pi N,\ N \in \mathbb{Z}. \tag{6.8}$$

Then for $T = J\frac{d}{dx} + V$ we have

$$T_W = \Psi^*(T+r)\,\Psi = \Psi^*\left(J\frac{d}{dx} + V(x) + r\right)\Psi = J\frac{d}{dx} + e^{-2bJ}\,V(x)$$

$$= J\frac{d}{dx} + W(x), \tag{6.9}$$

where $W = e^{-2bJ}V$ and $c = \cos 2b$, $s = \sin 2b$,

$$W \equiv \begin{pmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} q_1 & q_2 \\ q_2 & -q_1 \end{pmatrix} = \begin{pmatrix} cq_1 + sq_2 & -sq_1 + cq_2 \\ -sq_1 + cq_2 & -cq_1 - sq_2 \end{pmatrix}. \tag{6.10}$$

Then

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = e^{2bJ}q = \begin{pmatrix} cq_1 + sq_2 \\ -sq_1 + cq_2 \end{pmatrix}. \tag{6.11}$$

We have

$$T_W = \Psi^*(T + rI)\,\Psi, \qquad \sigma(T_W) = \sigma(T) + r,\ r = \pi N. \tag{6.12}$$

Consider the case $t = 0$. For $\psi(x, z, q)$ we have Eq. (2.5), and then

$$\psi(x, z, q) = \Psi(x)\,\psi(x, \zeta, w), \qquad (\Psi f)(x) = e^{rxJ}f(x),$$
$$M(z, q) = (-1)^N\,M(\zeta, w), \tag{6.13}$$

where $\zeta = z + r$. Let $k(z, q)$ be the quasimomentum for q. By the symmetry we obtain

$$k(\zeta, w) = r + k(\zeta - r, q), \qquad r = \pi N. \tag{6.14}$$

and then

$$k(\zeta, w) = \zeta + O(1/\zeta), \qquad |\zeta| \to \infty.$$

Using (6.13)–(6.14) we obtain

$$\varphi_1(1, z, q) = (-1)^N\,\varphi_1(1, \zeta, w), \qquad \varphi_2(1, z, q) = (-1)^N\,\varphi_2(1, \zeta, w). \tag{6.15}$$

Then

$$m_n(q) + r = m_{n+N}(w), \qquad z_n^{\pm}(q) + r = z_{n+N}^{\pm}(w),\ n \in \mathbb{Z}. \tag{6.16}$$

We get

$$h_{n+N,2}(w) = -\log|\varphi_2(1, m_{n+N}(w), w)| = -\log|\varphi_2(1, m_n(q), q)| = h_{n,2}(q)$$

and

$$\begin{aligned} h_{n+N,1}(w) &= |h_{n+N,1}(w)| \operatorname{sign}(z_{n+N}(w) - m_{n+N}(w)) \\ &= |h_{n,1}(q)| \operatorname{sign}(z_n(q) - m_n(q)) = h_{n,1}(q). \end{aligned}$$

Hence we have (1.8). ▮

Consider now the simple example of the inverse problem. Assume that only one gap $\gamma_0 = (-1, 1)$ is open and other gaps γ_n, $n \neq 0$, are closed. We consider the Dirac operator with $c = \cos 2t$, $s = \sin 2t$, $t \in [0, \pi)$ and we write the system in the form

$$J\varphi' + V\varphi = z\varphi, \qquad V = \begin{pmatrix} c & s \\ s & -c \end{pmatrix}, \ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{6.17}$$

where φ_1, φ_2 are functions in $t \in \mathbb{R}$. We find the solution of this system in the form $\psi = \exp(ikx)\,\psi_0$, where $\psi_0 \in \mathbb{C}^2$ is constant. Then

$$\det(V - zI + ikJ) = -k^2 - s^2 + z^2 - c^2 = 0,$$

and therefore the quasimomentum $k = \sqrt{z^2 - 1}$, and

$$\psi_\pm = e^{\pm ikx} \begin{pmatrix} 1 \\ a_\pm \end{pmatrix}, \qquad a_\pm = \frac{s \mp ik}{c + z}. \tag{6.18}$$

The fundamental solution $\varphi = C_1\psi_+ + C_2\psi_-$ and then

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \varphi(0) = C_1 \begin{pmatrix} 1 \\ a_+ \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ a_- \end{pmatrix}.$$

Therefore

$$C_1 + C_2 = 0, \qquad C_1(a_+ - a_-) = 1, \qquad C_1 = -C_2 = -\frac{c + z}{2ik},$$

and

$$\varphi = C_1\left(e^{ikx}\begin{pmatrix} 1 \\ a_+ \end{pmatrix} - e^{-ikx}\begin{pmatrix} 1 \\ a_- \end{pmatrix}\right) = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$

Then

$$\varphi_1(x,z)=C_1 2i\sin kx=-\frac{c+z}{k}\sin kx,\qquad \varphi_1(1,z)=-(c+z)\frac{\sin k}{k}. \quad (6.19)$$

$$\varphi_2(x,z)=\cos kx-s\frac{\sin kx}{k},\qquad \varphi_2(1,z)=\cos k-s\frac{\sin k}{k}. \quad (6.20)$$

The zero of $\varphi_1(1,z)$ coincides with the Dirichlet spectrum, i.e., $c+m_0=0$ and then the Dirichlet eigenvalue $m_0=-\cos 2t$ and $m_n=z_n^{\pm}=\sqrt{1+(\pi n)^2}\,\text{sign}\, n$, $n\neq 0$.

We have $k(m_0)=i\sqrt{1-c^2}=i\tau$ and $\tau\in\mathbb{R}$, $|\tau|=|s|$. Then

$$\varphi_2(1,m_0)=\cosh s-s\frac{-\sinh\tau}{i^2\tau}=e^{-s},\qquad h_{02}=s,$$
$$h_{01}=|c|\,\text{sign}\, c=c,\qquad \text{and}\qquad h_n=0,\ n\neq 0.$$

Then we get the solution of the inverse problem in the case of one gap.

Assume that $|\gamma_N|\neq 0$ and $\gamma_N=(-1+r,1+r)$, $m_N=-c+r$, $r=\pi N$ and assume other gaps γ_n, $n\neq N$, are closed. Then using (1.8) we obtain $h_{N2}=s$, $h_{N1}=c$, and $h_n=0$, $n\neq N$, and

$$\gamma_N=(z_N^-,z_N^+),\ \ldots,\ z_{N-1}^{\pm}=-\sqrt{1+\pi^2}+r,\qquad z_{N+1}^{\pm}=\sqrt{1+\pi^2}+r,\ \ldots$$

Then we get the solution of the inverse problem in the case of one gap:

$$V=\begin{pmatrix}\cos 2b & \sin 2b\\ \sin 2b & -\cos 2b\end{pmatrix},\qquad b=t-\pi N x.$$

ACKNOWLEDGMENTS

The author is grateful for financial support given by SFB 288.

REFERENCES

[GT1] J. Garnett and E. Trubowitz, Gaps and bands of one dimensional periodic Schrödinger operators, *Comment. Math. Helv.* **59** (1984), 258–312.

[GT2] J. Garnett and E. Trubowitz, Gaps and bands of one dimensional periodic Schrödinger operators, II, *Comment. Math. Helv.* **62** (1987), 18–37.

[KK1] P. Kargaev and E. Korotyaev, The inverse problem for the Hill operator, the direct approach, *Invent. Math.* **129**, No. 3 (1997), 567–593.

[KK2] P. Kargaev and E. Korotyaev, Effective masses and conformal mappings, *Commun. Math. Phys.* **169** (1995), 597–625.

[K1] E. Korotyaev, The inverse problem and the trace formula for the Hill operator, II, *Math. Z.* **231** (1999), 345–368.

[K2] E. Korotyaev, The inverse problem for the Hill operator, I, *Internat. Math. Res. Notices* **3** (1997), 113–125.

[K3] E. Korotyaev, Metric properties of conformal mappings on the complex plane with parallel slits, *Internat. Math. Res. Notices* **10** (1996), 493–503.

[K4] E. Korotyaev, The propagation of the waves in the one-dimensional periodic media, *Asymptotic Anal.*, **15**, 1–24,

[K5] E. Korotyaev, The estimate of periodic potentials in terms of effective masses, *Commun. Math. Phys.* **183** (1997), 383–400.

[K6] E. Korotyaev, Estimate for the Hill operator, I, *J. Differential Equations* **162** (2000), 1–26.

[K7] E. Korotyaev, Estimate of periodic potentials in terms of gap lengths, *Commun. Math. Phys.* **197**, No. 3 (1998), 521–526.

[K8] E. Korotyaev, Periodic weighted operator, preprint, SFB-288, 1999.

[K9] E. Korotyaev, Inverse problem and estimates for periodic Zakharov-Shabat systems, preprint, SFB-288, 1999.

[K10] E. Korotyaev, Inverse problem for periodic weighted operator, *J. Funct. Anal.* **170** (2000), 188–218.

[Kr] M. Krein, On the characteristic function $A(\lambda)$ of a linear canonical system differential equation of the second order with periodic coefficients, *Prikl. Mat. Mekh.* **21** (1957), 320–329. [In Russian]

[Le] N. Levinson, "Gap and Density Theorem," Amer. Math. Soc. Colloq. Publ., Amer. Math. Soc., New York, 1940.

[LS] B. Levitan and I. Sargsjan, "Sturm–Liouville and Dirac Operators," Mathematics and Its Applications (Soviet Series), Vol. 59, Kluwer Academic, Dorderecht, 1991.

[MO1] V. Marchenko and I. Ostrovski, A characterization of the spectrum of the Hill operator, *Math. USSR. Sb.* **26** (1975), 493–554.

[MO2] V. Marchenko and I. Ostrovski, Approximation of periodic by finite-zone potentials, *Selecta Math. Sov.* **6**, No. 2 (1987), 101–136.

[Mi1] T. Misura, Properties of the spectra of periodic and antiperiodic boundary value problems generated by Dirac operators, I, II, *Theoret. Funktsii Funkt. Anal. i Prilozhen*, **8** (1978), 90–101; **31** (1979), 102–109. [In Russian]

[Mi2] T. Misura, Finite-zone Dirac operators, *Theoret. Funktsii Funkt. Anal. i Prilozhen*, **33** (1980), 107–11.

[PT] P. Pöschel and E. Trubowitz, "Inverse Spectral Theory," Academic Press, Boston, 1987.

[T] E. Titchmarsh, "The Theory of Functions," 2nd ed., Univ. Press, London, 1975.